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Analytic properties of energy levels in models with delta-function potentials

A G Ushveridze

Institute of Physics of the Georgian Academy of Sciences, Guramishvili 6, 380077 Tbilisi, USSR

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Abstract. The analytic properties of energy levels as the functions of coupling parameters in one-dimensional quantum mechanical models described by the potentials of delta-function type are investigated. Here the following models are considered: (i) the rectangular potential well with delta-function barrier, (ii) two delta-function wells in the unbounded space, (iii) the Kronig-Penney model with periodic delta-function potential, and (iv) the Bose gas with delta-function interaction between particles.

1. Introduction

The study of analytic properties of the eigenvalues of linear differential operators is a very interesting problem of mathematical physics. Its importance is obvious: the location and character of spectral singularities determine the convergence properties of various perturbative expansions. The first physical model for which this problem has been studied in detail is a simple quartic anharmonic oscillator defined by the Schrödinger equation

$$[-\partial^2/\partial x^2+(x^2-g)^2]\psi(x,g)=E(g)\psi(x,g) \qquad \psi(\pm\infty,g)=0. \tag{1}$$

This model is of particular interest to field theoreticians because it can be treated as a scalar field theory in one-dimensional spacetime. Bender and Wu (1969), Simon (1970) and Shanley (1986) argued that the energy levels in (1) as the functions of a strong coupling parameter g have an infinite number of square-root type branch-point singularities whose physical origin is the crossing of the levels in the complex g plane. They demonstrated that the levels of equal parity form a common Riemann surface and thus can be obtained from each other by simple analytic continuation. Further investigations showed that these properties are not specific for the model (1) only (see, e.g., Bender and Happ 1974, Hunter and Guerrieri 1982).

Today it is not yet clear how the topology of a spectral Riemann surface depends on the type of model under consideration. Therefore the accumulation of results obtained for various concrete models seems at present to be very useful. Note, however, that models with polynomial potentials are not convenient for this purpose, since the study of their analytic properties requires the use of rather refined mathematical techniques. This is the main reason why there are so few publications on this topic.

Fortunately, there exists a wide class of models which are much simpler than the polynomial ones. They are described by potentials of delta-function type. The corresponding Schrödinger equations are equivalent to the systems of simple transcendental

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equations and therefore the analytic properties of their solutions can be studied without any difficulties. These models are interesting from the point of view of their field-theoretical generalisations. As an example we refer to the non-linear Schrödinger model which, being the non-relativistic field theory in two-dimensional spacetime, describes the system of bosons interacting via a two-body delta-function potential. The Bethe ansatz reduces the N-particle Schrödinger equations for this model to the system of N coupled transcendental equations (Lieb and Liniger 1963).

The present paper is organised as follows. In § 2 we consider the Z_2-symmetric rectangular double square-well potential in which the delta function plays the role of a potential barrier. Studying the analytic properties of energy levels as functions of delta-function 'height' we obtain the location of all spectral singularities which are found to be of square-root type. We propose a simple qualitative method enabling us to elucidate which of the levels cross in the singular points. In § 3 we investigate the large-order behaviour of perturbation expansions for this model and demonstrate that it strongly depends on the type of observable to be expanded. In § 4 we introduce a new parameter determining the position of the delta-function barrier and study the dependence of energy levels on this parameter. We show that the origin of the well known phenomenon of 'quasicrossing' of levels is the real crossing of these levels in a complex plane. The subject of § 5 is the model with two delta-function wells in the unbounded space. We demonstrate that though this model has only a finite number of bound states, the corresponding energy levels as functions of delta-function 'depth' have an infinite number of square-root type singularities. We argue that the physical nature of these singularities is the crossing of bound states with the quasistationary ones appearing when the 'depth' of the wells becomes negative. In § 6 we consider the case when the delta functions have different 'depths'. We demonstrate that the presence of two parameters in the problem makes possible the appearance of cube-root type singularities corresponding to the triple crossing of the levels. In § 7 we briefly discuss the Kronig-Penney model with the periodic delta-function potential and show that the fixation of quasimomenta makes the energy levels singular in the central points of the coupling constant plane. Section 8 is devoted to the non-linear Schrödinger model. We consider only the case of a finite number of Bose particles moving on the circle of finite length and demonstrate that an increase in the number of these particles complicates the analytic structure of the spectrum. In this work we do not intend to study the thermodynamic limit of this model. A subsequent paper will be devoted to this topic.

2. The double square-well potential: symmetric case

The first model which we discuss here is defined by the Schrödinger equation

$$[-\partial^2/\partial x^2+2g\delta(x)]\psi(x,g)=E(g)\psi(x,g) \qquad \psi(\pm 1,g)=0. \tag{2}$$

This can be treated as an idealisation of a polynomial double-well potential model (1). Here g plays the role of the strong coupling constant and determines (as in the case of (1)) the penetrability of the potential barrier. This model is well known in the literature (see, e.g., Flügge 1971). The eigenfunctions of (2) have the form

$$\psi(x,g)=\begin{cases} a_-\sin[k(g)(x+1)] & x<0 \\ a_+\sin[k(g)(x-1)] & x>0 \end{cases} \tag{3}$$

and the eigenvalues are

$$E(g) = k^2(g). \tag{4}$$

Connecting the left and right branches of (3) at $x=0$ by the boundary condition

$$\frac{\partial}{\partial x} \ln \psi(x, g) \Big|_{x=-0}^{x=+0} = 2g \tag{5}$$

and using (4) one obtains the transcendental equation for $E(g)$:

$$\sqrt{E(g)} \cot \sqrt{E(g)} = -g. \tag{6}$$

Note that (6) describes only the even part of the spectrum. The eigenfunctions of the odd states have nodes at $x=0$, and therefore the corresponding energy levels do not depend on g.

It is known that equation (6) has an infinite number of solutions for any real value of g. We denote these solutions by $E_n(g)$ assuming the usual order of their numeration

$$E_0(g) < E_1(g) < E_2(g) < \dots . \tag{7}$$

Now let us use equation (6) for the study of singularities of the levels $E_n(g)$ in the complex g plane. It is natural to expect that in the singular points g_s, the multiple-valued function $E(g)$ defined by (6) has infinite derivatives with respect to g. Then, differentiating (6) and using the condition

$$\frac{\partial E(g)}{\partial g}\Big|_{g=g_s} = \infty \tag{8}$$

we get the equation

$$2\sqrt{E_s} = \sin 2\sqrt{E_s} \tag{9}$$

which together with

$$\sqrt{E_s} \cot \sqrt{E_s} = -g_s \tag{10}$$

determines the positions of such singular points. Note that (9) coincides with the condition of vanishing of the normalisation integrals

$$\int_{-1}^{1} \psi^2(x, g)\, dx \sim \left(1 - \frac{\sin 2\sqrt{E(g)}}{2\sqrt{E(g)}}\right). \tag{11}$$

This fact is in full accordance with the analogous result which Bender and Wu (1969) and Simon (1970) obtained for the model (1).

It is not difficult to see that the system of equations (9) and (10) has an infinite number of pairs of complex-conjugated solutions. The first several of them can be obtained numerically. They are:

$$\begin{aligned}
g_s^{(1)} &= -3.790\,53 \pm 7.438\,96\mathrm{i} \\
g_s^{(2)} &= -4.360\,43 \pm 13.865\,95\mathrm{i} \\
g_s^{(3)} &= -4.721\,12 \pm 20.214\,8\mathrm{i} \\
g_s^{(4)} &= -4.985\,83 \pm 26.536\,6\mathrm{i} \\
g_s^{(5)} &= -5.195\,20 \pm 32.844\,5\mathrm{i}.
\end{aligned} \tag{12}$$

The chosen numeration of these solutions corresponds to the increase of their distance from the origin. The next singularities can be found by applying the iteration procedure to the system

$$E_s^{(n)} = (n\pi + \tfrac{1}{2}\sin^{-1} 2\sqrt{E_s^{(n)}})^2 \tag{13}$$

$$g_s^{(n)} = \cos^2 \sqrt{E_s^{(n)}} \tag{14}$$

which, evidently, is equivalent to (9) and (10). For large n the result (the analogue of the semiclassical result for the model (1)) is

$$g_s^{(n)} \approx -\ln n \pm i n\pi. \tag{15}$$

The fact that the branches of the $\sin^{-1}$ term in (13) are chosen correctly follows from the proximity of semiclassical solutions (15) to the numerical ones (12) for any n (even for $n=1$). We see that, as in the model (1), the sequence of singular points tends to infinity, but in our case they all lie in the left half-plane of g. This is the first difference between the two models.

In order to determine the character and physical nature of the obtained singularities, let us expand equation (6) near an arbitrary point g_s. Using (9) and (10) we conclude that

$$E(g) = E_s \pm \sqrt{E_s}\,(g-g_s)^{1/2} + \ldots \tag{16}$$

when $g \to g_s$. This means that we deal with the square-root type branch-point singularities and their origin is the crossing of two levels described by the two branches in (16). The result is not surprising. It reproduces the properties of model (1) discussed in § 1. It is worth noticing that expression (16) justifies the assumption (8) and thus makes the above-mentioned derivation self-consistent. Writing out the following corrections to (16), it can be easily understood that the general form of expansion of $E(g)$ near the singular point is

$$E(g) = \sum_{n=0}^{\infty} (g-g_s)^n A_n \pm (g-g_s)^{1/2} \sum_{n=0}^{\infty} (g-g_s)^n B_n. \tag{17}$$

In order to answer the main question of this section, i.e. which of the levels cross in the point $g_s^{(n)}$, we consider the path Γ_n shown in figure 1. Having begun at the point $g=0$, it passes round the singular point $g=g_s^{(n)}$ and then goes back to the initial point $g=0$. Due to the simplicity of equation (6), the analytic continuation of various energy levels along this path can be done numerically. By analysing the results obtained numerically it can be shown that the analytic continuation along the path Γ_n transforms two levels $E_0(g)$ and $E_n(g)$ into each other and does not change the level $E_m(g)$ if m differs from 0 and n. This fact, which has been verified for all $n \leq 10$, enables us to conclude that any singular point $g_s^{(n)}$ is the crossing point only for the levels $E_0(g)$ and $E_n(g)$. The existence of composite paths $\Gamma_{n_1} \times \Gamma_{n_2}$ transforming $E_{n_1}(g)$ into $E_{n_2}(g)$ for any n_1 and n_2 means that all the analytically continued functions form the common Riemann surface.

The topology of this surface is rather non-trivial. Indeed, consider the path S_n which begins at the point $g=0$, passes round the first n singular points $g_s^{(1)}, g_s^{(2)}, \ldots, g_s^{(n)}$ in the positive direction and returns to the initial point (see figure 1). Obviously, S_n is equivalent to the composite path $\Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n$ which, according to the previous reasoning, transforms $E_0(g)$ into $E_1(g)$, $E_1(g)$ into $E_2(g), \ldots$, and $E_n(g)$ into $E_0(g)$ again. Taking the limit $n \to \infty$ we see that the infinitely large path S_∞ plays the role of the 'creation operator' for model (2).

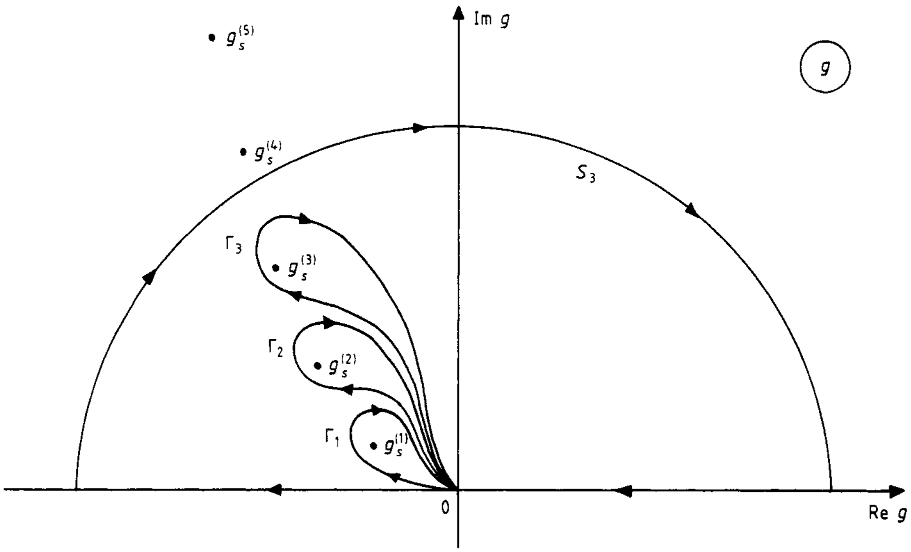

Figure 1. A qualitative picture of the location of singular points $g_s^{(n)}$ and of the corresponding paths Γ_n and S_n in the complex g plane (model (2)).

Thus we have seen that, though model (2) is similar to model (1) (in both cases we have two wells separated by the potential barrier), there exists a remarkable difference between the topologies of corresponding Riemann surfaces: in contrast to model (1), only the ground-state energy in (2) has an infinite number of pairs of singular points, at which it successively crosses other energy levels; each excited level has only one pair of singular points at which it crosses with the ground state.

Such a difference between the two models (1) and (2) can be explained as follows. Consider a sufficiently far area in the complex g plane (this may be the exterior of the large circle), and continue there the spectrum of the system under consideration. There exist at least two different ways of such a continuation. The first is based on the continuation from the large negative values of g and the second from the large positive ones. For model (2) the results are

$$E_n^{(-)}(g) \approx \begin{cases} n^2\pi^2 & n>0 \\ -g^2 & n=0 \end{cases} \qquad \text{from } g \approx -\infty \tag{18a}$$

and

$$E_m^{(+)}(g) \approx (m+1)^2\pi^2 \qquad m \geqslant 0 \qquad \text{from } g \approx +\infty \tag{18b}$$

respectively. The condition of coincidence of both asymptotics is

$$E_m^{(+)}(g) = E_n^{(-)}(g). \tag{19}$$

Formulae (18) and (19) imply that $E_n^{(+)}(g) = E_{n+1}^{(-)}(g)$ for any n. This result is not surprising. It expresses the above-mentioned fact that the arbitrary S_∞-type path has the meaning of the creation operator in (2). The more interesting case arises when $n=0$ and m is arbitrary. Then equation (19) allows a discrete set of non-trivial solutions

$$g_s^{(n)} \approx \pm \mathrm{i} n\pi \qquad n = 1, 2, \ldots \tag{20}$$

giving the location of the points at which the levels $E_0(g)$ and $E_n(g)$ cross each other. We see that for large n equation (20) reproduces result (15) obtained by solving equations (9) and (10).

Now let us consider model (1). In the case $g \approx -\infty$ the anharmonic term is small in comparison with the harmonic one and, hence, the spectrum of model (1) approaches the spectrum of the harmonic oscillator

$$E_n^{(-)}(g) \approx g^2 + n\sqrt{-2g} \qquad \text{from } g \approx -\infty. \tag{21a}$$

In the second case of $g \approx +\infty$ we have two well separated wells, each being described with a good accuracy by the harmonic potential. Therefore the spectrum of (1) is now

$$E_m^{(+)}(g) = m\sqrt{2g} \qquad \text{from } g \approx +\infty. \tag{21b}$$

Substituting (21a) and (21b) into equation (19) we obtain that it holds for any n and m if

$$g_s = (n + \mathrm{i}m)^{2/3}. \tag{22}$$

This means that each level in (1) crosses any other level and the crossing points lie in the complex-conjugated points of the right g half-plane, which is in full accordance with the results of Bender and Wu (1969) and Shanley (1986).

The proposed method does not provide an exact calculation of the positions of singular points. But it is a good tool for the qualitative analysis of topological properties of the spectral Riemann surface.

3. The properties of the 'strong coupling expansion'

Now let us discuss the convergence properties of perturbation expansions in powers of g. Since $g = 0$ is a regular point of the function $E_n(g)$, the expansion of $E_n(g)$ near this point has a non-zero radius of convergence. This radius (which we denote by $R\{E_n(g)\}$ is, evidently, equal to the distance to the nearest singularity of $E_n(g)$ in the complex g plane. From the previous reasoning it follows that

$$R\{E_0(g)\} = |g_s^{(1)}| \tag{23a}$$

and

$$R\{E_n(g)\} = |g_s^{(n)}| \qquad n = 1, 2, 3, \ldots. \tag{23b}$$

Remember now that the functions $E_0(g)$ and $E_n(g)$ cross at the point $g_s^{(n)}$. This means that their expansions near this point have the form (17) and, therefore, the sum of both functions $E_0(g)$ and $E_n(g)$ must be regular at $g = g_s^{(n)}$. Analogously, we can conclude that the sum $E_0(g) + E_1(g) + \ldots + E_n(g)$ is regular at all points $g_s^{(1)}$, $g_s^{(2)}, \ldots, g_s^{(n)}$, and hence, the nearest singularity of this sum determining the radius of convergence of its perturbation expansion turns out to be $g_s^{(n+1)}$:

$$R\{E_0(g) + \ldots + E_n(g)\} = |g_s^{(n+1)}|. \tag{24}$$

It is not difficult to see that the expression (24) allows the simple generalisation

$$R\left\{\sum_{k=0}^{n} f(E_k(g))\right\} = |g_s^{(n+1)}| \tag{25}$$

in which $f(z)$ is an arbitrary integral function. Substituting $f(z) = \exp(-\beta z)$ into (25) and taking the limit $n \to \infty$ we obtain that the statistical sum of model (2)

$$Q(\beta, g) = \sum_{n=0}^{\infty} \exp(-\beta E_n(g)) \tag{26}$$

is an integral function of g for any values of β. In other words, its expansion in powers of g has an infinite radius of convergence.

We emphasise that this result holds also for model (1). It can be immediately proved by calculating large orders of the perturbative strong-coupling expansions by means of the saddle-point approximation in the path integral approach.

How do the asymptotics of the perturbative series behave? To answer this question, consider any quantity of the type $A(g)=\Sigma_{k=0}^{n} f(E_k(g))$ and write the dispersion relation for the coefficients of its expansion in powers of g

$$A_n=\frac{1}{2\pi i}\oint\frac{dg}{g^{N+1}}A(g). \tag{27}$$

For large N the vicinities of nearest singularities of $A(g)$ make the main contribution to the integral (27). Since in models (1) and (2) they are located in the complex-conjugated points and are of square-root type, we obtain

$$A_N\approx(g_s^N c_0+g_s^{*N}c_0^*)N^{-3/2}\approx|g_s|^N N^{-3/2}\cos(N\varphi_s+\varphi_0) \tag{28}$$

where $\varphi_s=\arg(g_s)$. Thus, in contrast to the weak-coupling expansion in (1) in which the signs in the terms alternate, the strong-coupling expansion in both models (1) and (2) has an oscillative character. The oscillation frequencies are determined by the proximity of the singular points g_s and g_s^* to the real g axis.

4. The double square-well potential: general case

In this section we consider the generalised version of model (2) described by the Schrödinger equation

$$[-\partial^2/\partial x^2+2g\delta(x-\lambda)]\psi(x,\lambda)=E(\lambda)\psi(x,\lambda) \qquad \psi(\pm1,\lambda)=0. \tag{29}$$

Here λ is the asymmetry parameter determining the position of the delta-function potential barrier. In contrast to the previous case we assume g to be fixed and study the dependence of energy levels on λ. The corresponding spectral equation (the generalisation of equation (6)) is

$$\sqrt{E(\lambda)}\,\{\cot[\sqrt{E(\lambda)}(1+\lambda)]+\cot[\sqrt{E(\lambda)}(1-\lambda)]\}=-2g. \tag{30}$$

If $\lambda=0$, equation (30) transforms into (6).

If $g=\infty$, the delta-function barrier becomes unpenetrable and we obtain two quite disconnected wells. We denote the spectra in the left and right wells by $E_n^{(-1)}(\lambda)$ and $E_n^{(+1)}(\lambda)$, respectively. We have

$$E_n^{(\sigma)}(\lambda)=\pi^2 n^2/(1-\sigma\lambda)^2 \tag{31}$$

where $n=0,1,2,\ldots$ and $\sigma=\pm1$. We see that the levels $E_n^{(-)}(\lambda)$ and $E_m^{(+1)}(\lambda)$ coincide at the points $\lambda^{(nm)}=(n-m)/(n+m)$, i.e.

$$E_n^{(-1)}(\lambda^{(nm)})=E_m^{(+1)}(\lambda^{(nm)})=\tfrac{1}{4}\pi^2(n+m)^2. \tag{32}$$

But this is not the real crossing since the corresponding two states relate to different Hilbert spaces.

Now let us consider the more interesting case of $g<\infty$. If g is large, the penetration through the barrier is suppressed (it is of order g^{-1}) and therefore we can use the previous classification of levels by means of quantum numbers n and σ. The dependence of $E_n^{(\sigma)}(\lambda)$ on the 'physical' values of λ (in the interval $[-1, 1]$) is shown in figure 2. We see that near the points $\lambda^{(nm)}$ the levels $E_n^{(-)}(\lambda)$ and $E_m^{(+1)}(\lambda)$ become close, $|E_n^{(-1)}(\lambda)-E_m^{(+1)}(\lambda)|\sim g^{-1}$, and exchange their quantum numbers without crossing. This phenomenon is known under the name of 'quasicrossing' of levels. The anomalous proximity of levels at $\lambda=\lambda^{(nm)}$ may be an indication of the existence of crossing points, $\lambda_s^{(nm)}$, in the complex λ plane near the real axis, $|\lambda^{(nm)}-\lambda_s^{(nm)}|\sim g^{-1}$.

To verify this assumption, let us take

$$\lambda=\frac{n-m}{n+m}+\frac{1}{g}\mu \tag{33}$$

and

$$E(\lambda)=\tfrac{1}{4}\pi^2(n+m)^2(1+g^{-1}e(\mu)+O(g^{-2})). \tag{34}$$

Substituting (33) and (34) into (30) and neglecting the terms of order g^0 we obtain the equation for $e(\mu)$

$$\left(\frac{ne(\mu)}{n+m}+\mu\right)^{-1}+\left(\frac{me(\mu)}{n+m}-\mu\right)^{-1}=-2 \tag{35}$$

from which it follows that

$$e(\mu)=\frac{(n+m)^2}{2nm}\left\{-\left(\frac{1}{2}+\frac{m-n}{m+n}\mu\right)\pm\left[\left(\frac{1}{2}+\frac{m-n}{m+n}\mu\right)^2+\frac{4nm}{(n+m)^2}\mu^2\right]^{1/2}\right\}. \tag{36}$$

We see from (36) and (34) that $E(\lambda)$ has square-root type branch points. Their locations are given by the condition of vanishing of the root in (36):

$$\lambda_s^{(nm)}=\frac{n-m}{n+m}+\frac{1}{g}\left(\frac{n-m}{2(n+m)}\pm\mathrm{i}\frac{\sqrt{nm}}{n+m}\right). \tag{37}$$

From this consideration it follows, in particular, that the physical nature of tunnel splitting of the levels in the symmetric case ($\lambda=0$) is the crossing of three levels in

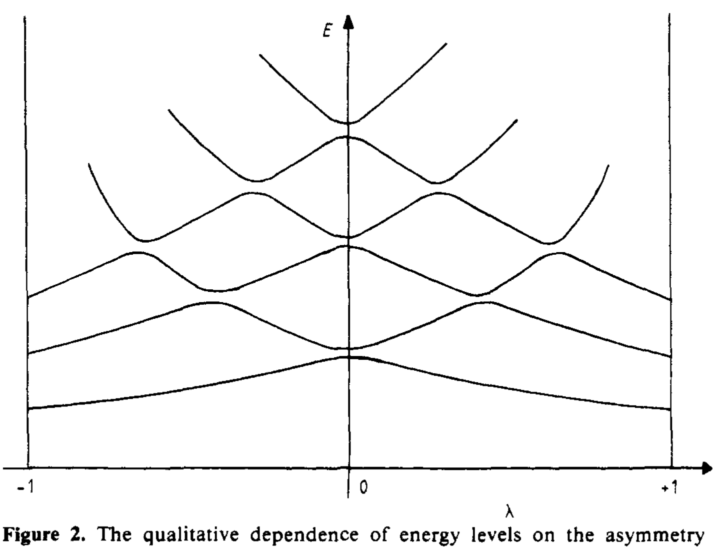

Figure 2. The qualitative dependence of energy levels on the asymmetry parameter λ (model (29)).

the complex λ plane in the vicinity of the origin. It is essential that λ is the parameter which breaks the Z_2 symmetry of the model.

An analogous explanation of the splitting and quasicrossing phenomena can be given for other models of this sort, e.g., for model (1) perturbed by any Z_2-non-invariant term.

5. Crossing with the quasistationary states

Until now we have investigated quantum systems in the finite volume. Here we consider two models defined in the unbounded space. We describe these models by the following Schrödinger equations:

$$[-\partial^2/\partial x^2 - 2g\delta(x)]\psi(x, g) = E(g)\psi(x, g) \tag{38}$$

and

$$[-\partial^2/\partial x^2 - g\delta(x-1) - g\delta(x+1)]\psi(x, g) = E(g)\psi(x, g). \tag{39}$$

We see that besides the continuous spectrum the system described by (38) and (39) allows a finite number of discrete energy levels at positive values of g. (If g is negative, these models do not have any bound states.)

If $g > 0$, model (38) has a single bound state with energy

$$E_0(g) = -g^2. \tag{40}$$

The fact that this level is regular everywhere follows from the observation that other discrete levels with which it could possibly cross do not exist.

Now consider model (39). If $g > 0$, it has only two bound states with the energies $E_0^{(-1)}(g)$ and $E_0^{(+1)}(g)$. They can be obtained from the simple transcendental equation

$$\sqrt{-E_0^{(\sigma)}(g)}\,[1 + (\tanh\sqrt{-E_0^{(\sigma)}(g)})^\sigma] = g \qquad \sigma = \pm 1. \tag{41}$$

These levels cannot cross each other since the corresponding wavefunctions have opposed parities and thus relate to different invariant Hilbert subspaces. Hence, repeating the previous reasoning, it could be expected that each above-mentioned level would be regular everywhere. But this is not true! Analysing equation (41) we discover that, as in the case of model (2), the levels $E_0^{(\sigma)}(g)$ have an infinite number of singularities located in the points

$$g_s^{(\sigma)} = \pm 2\pi n\mathrm{i} + \ln(2\pi n\mathrm{i}) + \dots. \tag{42}$$

What is the cause of this phenomenon? To answer this question let us apply the qualitative method proposed in § 2 to model (39).

Following this method we find that for large positive values of g there exists only one level with the given parity

$$E_0^{(\sigma)}(g) \approx -g^2. \tag{43}$$

But for large negative values of g instead of the wells we have two delta-function barriers and, hence, an infinite number of quasistationary states between them. Neglecting the width of the levels we can write

$$E_n^{(\sigma)}(g) \approx (2\pi n)^2 \qquad n = 1, 2, \dots. \tag{44}$$

Substituting (43) and (44) into equation (19) we obtain the locations of the crossing points

$$g_s^{(n)} \approx \pm 2\pi n \mathrm{i} \tag{45}$$

which reproduce the result (42) with sufficient accuracy. From the coincidence of both results it follows that the physical origin of the infinite number of singularities is the crossing of the bound-state levels with the quasistationary ones appearing after the change of the sign of g.

6. Triple crossing

In this section we discuss the model defined by the following Schrödinger equation:

$$[-\partial^2/\partial x^2 - \tfrac{1}{2}g(1-\lambda)\delta(x+1) - \tfrac{1}{2}g(1+\lambda)\delta(x-1)]\psi(x,g) = E(g)\psi(x,g). \tag{46}$$

Now the energy levels depend on two parameters g and λ. We demonstrate that the presence of the second parameter λ in the problem makes possible the existence of cube-root type branch-point singularities in the complex g plane. These singularities can be treated as the points in which the triple crossing of energy levels occurs.

In order to obtain the locations of all such points, let us consider the spectral transcendental equation for (46)

$$\exp(-4\sqrt{-E(g)}) = \left(\frac{4\sqrt{-E(g)}}{g(1-\lambda)} - 1\right)\left(\frac{4\sqrt{-E(g)}}{g(1+\lambda)} - 1\right). \tag{47}$$

Introducing a new variable

$$z(g) = -4\sqrt{-E(g)}/g \tag{48}$$

we can reduce equation (47) to the more convenient form

$$g = \frac{1}{z(g)}\left[2\pi n \mathrm{i} + \ln\left(\frac{z(g)}{1-\lambda} + 1\right) + \ln\left(\frac{z(g)}{1+\lambda} + 1\right)\right]. \tag{49}$$

Let g_s be a certain point in the complex g plane. Substituting

$$g = g_s + \delta \tag{50a}$$

and

$$z(g) = z(g_s) + \varepsilon \tag{50b}$$

into (49) and expanding the right-hand side of (49) in powers of ε, we obtain

$$\begin{aligned} g_s + \delta = f_0(z_s) &+ \varepsilon\left(f_1(z_s) - \frac{f_0(z_s)}{z_s}\right) - \varepsilon^2\left(f_2(z_s) + \frac{f_1(z_s)}{z_s} - \frac{f_0(z_s)}{z_s^2}\right) \\ &+ \varepsilon^3\left(f_3(z_s) + \frac{f_2(z_s)}{z_s} + \frac{f_1(z_s)}{z_s^2} - \frac{f_0(z_s)}{z_s^3}\right) + \ldots \end{aligned} \tag{51}$$

where

$$f_0(z) = z^{-1}\left[2\pi n \mathrm{i} + \ln\left(\frac{z}{1-\lambda} + 1\right) + \ln\left(\frac{z}{1+\lambda} + 1\right)\right] \tag{52a}$$

$$f_1(z) = z^{-1}[(z+1-\lambda)^{-1} + (z+1+\lambda)^{-1}] \tag{52b}$$

$$f_2(z) = (2z)^{-1}[(z+1-\lambda)^{-2} + (z+1+\lambda)^{-2}] \tag{52c}$$

$$f_3(z) = (3z)^{-1}[(z+1-\lambda)^{-3} + (z+1+\lambda)^{-3}] \tag{52d}$$

and

$$z_s = z(g_s). \tag{53}$$

Now let

$$f_0(z_s) = g_s \tag{54a}$$

$$f_1(z_s) - f_0(z_s)/z_s = 0 \tag{54b}$$

$$f_2(z_s) + f_1(z_s)/z_s - f_0(z_s)/z_s^2 = 0. \tag{54c}$$

In this case from (51) and (54) it follows that

$$\varepsilon = [f(z_s)]^{1/3}\delta^{1/3} \tag{55}$$

and hence we deal with the cube-root type branch-point singularity whose physical origin is the crossing of three levels described by the three branches in (55). The values of the auxiliary parameter for which the triple points exist, the coordinates of these points and the corresponding values of the crossed energy levels can be obtained from system (54). After some simplifications we obtain

$$\lambda_s = \pm \mathrm{i}(z_s + 1) \tag{56a}$$

$$g_s = z_s^{-1}(1 + z_s)^{-1} \tag{56b}$$

where z_s satisfies the equation

$$(z_s + 1)^{-1} = 2\pi n \mathrm{i} + \ln \frac{2}{(z_s + 1)^{-2} + 1}. \tag{57}$$

Solving this equation in the large $|n|$ limit we obtain

$$\lambda_s = \pm\left(\frac{1}{2\pi n} - \frac{\mathrm{i}}{2\pi^2 n^2}\ln(\pi n)\right) \tag{58}$$

and

$$g_s = 2\pi n\mathrm{i} - 2\ln(2\pi n\mathrm{i}). \tag{59}$$

7. The Kronig–Penney model

The results of the previous sections can be generalised to the models with periodic potentials. The simplest model of this sort is the well known Kronig–Penney model described by the following Schrödinger equation:

$$\left(-\frac{\partial^2}{\partial x^2} + g\sum_{n=-\infty}^{\infty}\delta(x - n)\right)\psi(x, g) = E(g)\psi(x, g). \tag{60}$$

The spectrum of (60) has a band structure. It can be characterised by two quantum numbers: the number of the band and the quasimomentum K. If K is fixed, the spectrum becomes discrete. The corresponding spectral equation is

$$\sqrt{E(g)}\,[\cot\sqrt{E(g)} - \cos K/\sin\sqrt{E(g)}] = -g. \tag{61}$$

In the particular cases when $K = 0, \pm\pi/2$ or $\pm\pi$, (61) assumes an especially simple form

$$\sqrt{E(g)}\cot\sqrt{E(g)} = -g \tag{62a}$$

$$\sqrt{E(g)}\cot(\sqrt{E(g)}/2) = -g \tag{62b}$$

$$\sqrt{E(g)}\tan(\sqrt{E(g)}/2) = g. \tag{62c}$$

Equations ($62b$) and ($62c$) determine the edges of the bands. The asymptotic properties of the $E(g)$ functions can be investigated by means of the methods described in § 2. It is not difficult to show that these functions have an infinite number of singularities located in the points

$$g_s^{(n)} = \pm \mathrm{i} n\pi - \ln(\pi n) + \dots \qquad K = \pm\pi/2 \tag{63a}$$

$$g_s^{(n)} = \pm 2\mathrm{i} n\pi - 2\ln(\pi n) + \dots \qquad K = 0 \tag{63b}$$

$$g_s^{(n)} = \pm \tfrac{1}{2}\mathrm{i} n\pi - \tfrac{1}{2}\ln(\pi n) + \dots \qquad K = \pm\pi. \tag{63c}$$

The physical meaning of these singularities is the crossing of the levels relating to different bands but having the same quasimomenta K. This result can be easily generalised for arbitrary values of K.

Note that the analytic properties of energy levels in model (60) as functions of quasimomentum K have been considered in many works, see, e.g., the old work of Kohn (1959) and the paper of Avron and Simon (1978). We refer also to the paper of Ferrari *et al* (1985) presenting the numerical analysis of the Stark–Wannier metastable states in finite Kronig–Penney crystals extended to the complex field plane.

8. The interacting Bose gas

In this section we discuss the model of a Bose gas in which the interaction between bosons is determined by the two-body potential of the delta-function type. This model is equivalent to the non-linear Schrödinger model which is known to be an exactly integrable two-dimensional non-relativistic field theory (Thacker 1980). We consider the case of $N+1$ particles moving in the circle of unit length and assume N to be finite. We do not intend to discuss the thermodynamic limit of this theory, which will be considered in a separate work. We show that the analytic properties of energy levels in the finite N case can be investigated by using the methods described in the previous sections.

The Schrödinger equation for this model is

$$\left(-\sum_{0\le i\le N}\frac{\partial^2}{\partial x_i^2} + g\sum_{0\le i<k\le N}\delta(x_i-x_k)\right)\psi(x_0,\dots,x_N,g) = E(g)\psi(x_0,\dots,x_N,g). \tag{64}$$

The wavefunctions in (64) are symmetric and satisfy the periodic boundary conditions. The Bethe ansatz reduces the differential equation (64) to the system of $N+1$ coupled transcendental equations

$$\prod_{m\neq n}\frac{k_n(g)-k_m(g)-\mathrm{i}g}{k_n(g)-k_m(g)-\mathrm{i}g} = \exp(\mathrm{i}k_n(g)) \qquad n = 0,\dots,N. \tag{65}$$

Their solutions determine the energies of the system

$$E(g) = \sum_{n=0}^{N} k_n^2(g) \tag{66}$$

and the total momenta

$$P(g) = \sum_{n=0}^{N} k_n(g) \tag{67}$$

(Lieb and Liniger 1963). Introducing new variables

$$z_n(g) = \mathrm{i}k_n(g)/g \tag{68}$$

we can rewrite equations (65) and (66) in a more convenient form

$$gz_n(g) = 2\pi k_n\,\mathrm{i} + \sum_{m\neq n} \ln\left(\frac{z_n(g) - z_m(g) - 1}{z_n(g) - z_m(g) + 1}\right) \tag{69}$$

and

$$E(g) = -g^2 \sum_n z_n^2(g). \tag{70}$$

Repeating the reasoning of § 2 we conclude that the condition that the point g_s is singular has the form

$$\left.\frac{\partial E(g)}{\partial g}\right|_{g=g_s} = \infty. \tag{71}$$

Differentiating (70), we obtain

$$\frac{\partial E(g)}{\partial g} = -2g \sum_n z_n^2(g) - g^2 \sum_n z_n(g) \frac{\partial z_n(g)}{\partial g}. \tag{72}$$

Analogously, it follows from (69) that

$$\sum_m A_{nm}(g) \frac{\partial z_m(g)}{\partial g} = z_n(g). \tag{73}$$

Here

$$A_{nn}(g) = \sum_{m\neq n} [z_n(g) - z_m(g)]^{-2} - g \tag{74}$$

and

$$A_{nm}(g) = -[z_n(g) - z_m(g)]^{-2} \tag{75}$$

if $n \neq m$. Solving the matrix equation (73) and substituting the result into (72), we obtain the relation

$$\frac{\partial E(g)}{\partial g} = -2g \sum_n z_n^2(g) - g^2 \sum_{n,m} z_n(g) A_{nm}^{-1}(g) z_m(g). \tag{76}$$

According to (71), if $g = g_s$, the left-hand side of (76) is equal to infinity. For finite values of $z_n(g)$ the right-hand side of (76) may be infinite if and only if the matrix $A_{nm}(g)$ is singular, i.e. if the determinant of $A_{nm}(g)$ is zero. Hence, the condition that the point g_s is singular can be written as

$$\det\|A_{nm}(g)\|_{g=g_s}$$
$$\equiv \det\left\| -(z_n(g) - z_m(g))^{-2} + \delta_{nm} \sum_{i\neq k} (z_i(g) - z_k(g))^{-2} - \delta_{nm} g \right\|_{g=g_s} = 0. \tag{77}$$

This equation plays the central role in this section. Indeed, (77) together with Bethe ansatz equations (69) forms the complete system which enables us to determine the locations of all singular points g_s.

Korepin (1982) has shown that the norms of Bethe vectors are proportional to $\det\|A_{nm}(g)\|$. This means that instead of (77) one can use the condition

$$\int \psi^2(x_0, \ldots, x_N, g)\, \mathrm{d}x_0, \ldots, \mathrm{d}x_N \bigg|_{g=g_s} = 0 \tag{78}$$

in full accordance with the analogous result obtained by Bender and Wu (1969) and Simon (1970) for the case of model (1).

Now let us try to find the ground-state energy singularities. We confine ourselves to the investigation of those singularities which are located sufficiently far from the origin in the left half-plane of g. It is known (Thacker 1980) that for infinitely large real negative values of g the ground state of the system is determined by the so-called $(N+1)$-string configuration of the 'momentum' space

$$z_n(g) = N/2 - n \qquad g \approx -\infty. \tag{79}$$

Therefore it is reasonable to search the solutions of equations (69) and (77) in the form

$$z_n(g) = N/2 - n + \varepsilon_n(g) \tag{80}$$

assuming $\varepsilon_n(g)$ to be small if $\operatorname{Re} g \to -\infty$. Solving system (69) with respect to differences $\varepsilon_n(g) - \varepsilon_{n+1}(g)$ and preserving only the leading terms, we obtain

$$\varepsilon_n(g) - \varepsilon_{n+1}(g) = c(n) \exp[g(N-n)(n+1)/2] \qquad n = 0, 1, \ldots, N-1. \tag{81}$$

Here

$$c(n) = \prod_{k=N-n}^{N} A(k) \left(\prod_{k=1}^{n} A(k) \right)^{-1} \tag{82}$$

and

$$A(1) = 1 \qquad A(k) = \prod_{i=0}^{k-2} \frac{i-k-1}{i-k+1} \qquad k > 1. \tag{83}$$

Equation (77) in the same approximation assumes the form

$$\det\left(\sum_{n=0}^{N-1} (\varepsilon_n - \varepsilon_{n+1})^{-1} \hat{\Gamma}_n - g\hat{I} \right) = 0. \tag{84}$$

Here $\hat{\Gamma}_n$ and $\hat{I}$ are the $(N+1)\times(N+1)$ matrices

$$(\hat{I})_{ik} = \delta_{ik} \tag{85a}$$

and

$$\hat{\Gamma}_n = \delta_{in}\delta_{kn} + \delta_{i,n+1}\delta_{k,n+1} - \delta_{in}\delta_{k,n+1} - \delta_{kn}\delta_{i,n+1}. \tag{85b}$$

Substituting (81) into (84) we obtain the equation containing only the g variable

$$\det(\hat{\Lambda}(g) - g\hat{I})|_{g=g_s} = 0. \tag{86}$$

Here

$$\hat{\Lambda}(g) = \sum_{n=0}^{N-1} c^{-1}(n) \exp[-g(N-n)(n+1)/2]\hat{\Gamma}_n. \tag{87}$$

It is not difficult to see that $g_s = 0$ satisfies equation (86). But this solution is not that of the initial problem since the derivation of equation (86) assumes g_s to be sufficiently large. Denote by $\lambda_n(g)$, $n=1,2,\ldots,N$, the non-zero eigenvalues of the matrix $\hat{\Lambda}(g)$. Then equation (86) can be reduced to the system of N disconnected transcendental equations

$$g_s=\lambda_n(g_s) \qquad n=1,\ldots,N. \tag{88}$$

Now we consider concrete examples.

(i) $N=1$ (two-particle case). In this case

$$\lambda_1(g)=\tfrac{1}{2}e^{-g/2}. \tag{89}$$

Solving this equation, we obtain

$$g_s^{(k)}=4\pi k\mathrm{i}-2\ln(8\pi k\mathrm{i})+\ldots \qquad k\to\infty. \tag{90}$$

(ii) $N=2$ (three-particle case). In this case

$$\lambda_1(g)=\tfrac{1}{6}e^{-g} \tag{91}$$

$$\lambda_2(g)=\tfrac{1}{2}e^{-g} \tag{92}$$

for which we have two series of solutions

$$g_s^{(k)}=2\pi k\mathrm{i}-\ln(12\pi k\mathrm{i})+\ldots \tag{93}$$

$$g_s^{(k)}=2\pi k\mathrm{i}-\ln(4\pi k\mathrm{i})+\ldots \qquad k\to\infty. \tag{94}$$

(iii) $N=3$ (four-particle case). In this case

$$\lambda_1(g)=\tfrac{1}{6}e^{-3g/2} \tag{95}$$

$$\lambda_2(g)=(\tfrac{1}{12}e^{-3g/2}+\tfrac{1}{36}e^{-2g})+[(\tfrac{1}{12}e^{-3g/2})^2+(\tfrac{1}{36}e^{-2g})^2]^{1/2} \tag{96}$$

$$\lambda_3(g)=(\tfrac{1}{12}e^{-3g/2}+\tfrac{1}{36}e^{-2g})-[(\tfrac{1}{12}e^{-3g/2})^2+(\tfrac{1}{36}e^{-2g})^2]^{1/2} \tag{97}$$

and thus we have

$$g_s^{(k)}=\tfrac{4}{3}\pi k\mathrm{i}-\tfrac{2}{3}\ln(8\pi k\mathrm{i})+\ldots \tag{98}$$

$$g_s^{(k)}=\pi k\mathrm{i}-\tfrac{1}{2}\ln(18\pi k\mathrm{i})+\ldots \qquad k\to\infty. \tag{99}$$

$$g_s^{(k)}=\tfrac{4}{3}\pi k\mathrm{i}-\tfrac{2}{3}\ln(16\pi k\mathrm{i})+\ldots \tag{100}$$

Thus we see that the ground-state energy in model (64) in the $(N+1)$-particle case has N infinite series of singularities in the complex g plane. The physical origin of these singularities is the crossing of the ground-state energy with excited energy levels. This fact can be proved by means of the qualitative method described in § 2. A more detailed analysis of this model from the point of view of its analytic structure will be presented in a separate work.

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